Geometric Quantities Characterizing Wireless Power Transfer Between a Resonator and Resonant Dipoles

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***Abstract*—In this paper, the efficiency is calculated for power transfer between a large resonator and a collection of small resonant dipole receivers by means of either electric or magnetic dipole coupling. The efficiency is found to be a function of certain geometric quantities. These quantities may be used to characterize the effectiveness of various structures and materials for the purpose of wireless power transfer.**

***Index Terms*—efficiency, geometry, resonant**

I. Power Transfer to a Resonant Dipole

Consider a charge and current distribution that oscillates periodically in time, with period, $T = 2\pi/\omega$. Let it be assumed that the charge and current distribution is compact, and is embedded in an ambient electric field, $\vec{E}(x,t)$, and an ambient magnetic field, $\vec{B}(x,t)$. Let it further be assumed that both fields oscillate periodically with the same period, T, and that the ambient fields may be treated as approximately uniform over the characteristic length scale of the charge and current distribution. Let $\vec{E}(t)$ and $\vec{B}(t)$ represent the values of the ambient electric and magnetic fields evaluated at the central position of the charge and current distribution. The total work, W, done on the charge and current distribution by the ambient fields during one period is:

$$W = \int_0^T dt \, \left(\vec{E}(t) \cdot \dot{\vec{p}}(t) + \vec{B}(t) \cdot \dot{\vec{m}}(t) \right) \qquad (1)$$

where $\vec{p}(t)$ and $\vec{m}(t)$ are the instantaneous electric and magnetic dipole moments of the charge and current distribution.

If the oscillations are sinusoidal, the average power transferred to the charge and current distribution can be written in terms of products of vectors with complex phasor amplitudes:

$$P_{\text{trans}} = \tfrac{1}{2}\omega \, \Re \left[j \, \vec{B}^* \cdot \vec{m} \right] + \tfrac{1}{2}\omega \, \Re \left[j \, \vec{E}^* \cdot \vec{m} \right] \qquad (2)$$

It is clear that the transferred power is maximized when the amplitudes of $\vec{m}$ and $\vec{p}$ are maximized and shifted in phase from $\vec{B}$ and $\vec{E}$ by $-90°$. These conditions occur when a dipole has a resonance at the driving frequency. At resonance, the transferred power is:

$$P_{\text{trans}} = \tfrac{1}{2}\omega |\vec{B}||\vec{m}|\alpha_m + \tfrac{1}{2}\omega |\vec{E}||\vec{p}|\alpha_p \qquad (3)$$

where α_m and α_p are dimensionless alignment factors, given by:

$$\alpha_m \equiv |cos\theta_m| \qquad (4)$$

$$\alpha_p \equiv |cos\theta_p| \qquad (5)$$

and where θ_m is the angle between the magnetic field vector and the magnetic dipole moment vector, and θ_p is the angle between the electric field vector and the electric dipole moment vector.

II. Quality Factors and Dissipation

Let it be assumed that both the source and the receiver consist of structures which resonate at the same frequency, and that the source is being driven at its resonant frequency. The dipole receiver will oscillate in response to the mutual coupling between it and the source. Let $P_{\text{diss-s}}$ and $P_{\text{diss-r}}$ denote the power dissipated through undesired loss mechanisms in the source and receiver resonators respectively. The quality factors of the resonators are defined by the relations:

$$Q_s \equiv \frac{\omega \mathscr{E}_s}{P_{\text{diss-s}}}$$

$$Q_r \equiv \frac{\omega \mathscr{E}_r}{P_{\text{diss-r}}} \qquad (6)$$

where ω is the angular frequency of the oscillation, and $\mathscr{E}_s$ and $\mathscr{E}_r$ denote the stored energies in the source and receiver resonators, respectively.

In addition to the undesired dissipation, let it also be assumed that some power from the receiver resonator, P_{load}, is delivered to a load. This load will have an effective quality factor, Q_{eff}, given by:

$$Q_{\text{eff}} \equiv \frac{\omega \mathscr{E}_r}{P_{\text{load}}} \qquad (7)$$

and the dipole resonator will have an loaded quality factor, Q_{L}, given by:

$$\frac{1}{Q_{\text{L}}} = \frac{1}{Q_r} + \frac{1}{Q_{\text{eff}}} \qquad (8)$$

III. Definition of Dipole Volume

Because the dipole moment of the receiver is linearly proportional to its amplitude of oscillation, while its stored energy is proportional to the square of this amplitude, a constant may be defined by taking the ratio of the square of the dipole moment to the stored energy. In the case of a magnetic dipole, this constant is defined to be:

$$\upsilon \equiv \frac{\mu |\vec{m}|^2}{\mathscr{E}_r} \qquad (9)$$

while in the case of an electric dipole, it is defined to be:

$$\upsilon \equiv \frac{|\vec{p}|^2}{\epsilon \, \mathscr{E}_r} \qquad (10)$$

where μ and ϵ are the permeability and the permittivity of the medium surrounding the dipoles. The constant, υ, has units of volume, and characterizes the strength of the dipole moment created by the dipole resonator per unit of stored energy. Let this quantity be called the dipole volume.

It can be proven that the theoretical maximum dipole volume, υ_{max}, is $9\mathcal{V}_e$, where $\mathcal{V}_e$ is the volume of the smallest sphere which can completely enclose the resonator. (See Appendix D.) This theoretical upper bound allows an additional quantity, called the geometric efficiency to be defined:

$$\eta_g \equiv \frac{\upsilon}{\upsilon_{\text{max}}} \tag{11}$$

IV. Power Density, Capture Volume, and Power Transfer

The energy density, u, stored in the field of the source resonator is given by,

$$u = \frac{1}{2\mu}|\vec{B}|^2 \tag{12}$$

if it is a magnetic resonator, and by,

$$u = \frac{1}{2}\epsilon|\vec{E}|^2 \tag{13}$$

if it is an electric resonator. For either type of resonator, we may define the normalized energy density field, ρ, to be:

$$\rho \equiv \frac{u}{\mathscr{E}_s} \tag{14}$$

If the resonator is purely electromagnetic, i.e. all of its energy is stored in the electromagnetic field in the space surrounding the resonator, then ρ has the property:

$$\int d\mathcal{V}\ \rho = 1 \tag{15}$$

If we substitute these newly-defined quantities back into Equation 3, we get the following equation for the transferred power:

$$P_{\text{trans}} = \frac{\sqrt{2}}{2}\omega\alpha\sqrt{\mathscr{E}_s\mathscr{E}_r}\sqrt{\upsilon\rho} \tag{16}$$

which is true either for entirely electric coupling or for entirely magnetic coupling.

Let the dimensionless coupling coefficient, κ, be defined as:

$$\kappa \equiv \frac{P_{\text{trans}}}{\omega\sqrt{\mathscr{E}_s\mathscr{E}_r}} \tag{17}$$

In terms of υ and ρ, κ is found to be:

$$\kappa = \frac{\sqrt{2}}{2}\alpha\sqrt{\upsilon\rho} \tag{18}$$

Using the conservation of energy, and the assumption of steady state power transfer, the response of the resonant dipole to the ambient field may be calculated:

$$P_{\text{trans}} = \kappa\,\omega\sqrt{\mathscr{E}_s\mathscr{E}_r} = P_{\text{diss-r}} + P_{\text{load}} = \frac{\omega\mathscr{E}_r}{Q_{\text{L}}}$$
$$\sqrt{\frac{\mathscr{E}_r}{\mathscr{E}_s}} = \kappa Q_{\text{L}} \tag{19}$$

Let us define the receiver efficiency, η_r, to be the fraction of the transferred power which is delivered to the load:

$$\eta_r \equiv \frac{P_{\text{load}}}{P_{\text{trans}}} = \frac{\frac{\omega\mathscr{E}_r}{Q_{\text{eff}}}}{\frac{\omega\mathscr{E}_r}{Q_{\text{L}}}} = \frac{\frac{1}{Q_{\text{L}}} - \frac{1}{Q_{\text{r}}}}{\frac{1}{Q_L}} \tag{20}$$

From this definition, we get the following relation between Q_r, Q_{L}, and η_r:

$$Q_{\text{L}} = (1-\eta_r)Q_r \tag{21}$$

From Equations 19, 20, and 21, the power delivered to the load may be calculated as follows:

$$\begin{aligned} P_{\text{load}} &= \eta_r P_{\text{trans}} = \eta_r\omega\frac{1}{Q_{\text{L}}}\mathscr{E}_r = \eta_r\omega\kappa^2 Q_{\text{L}}\mathscr{E}_s \\ &= \eta_r(1-\eta_r)Q_r\omega\tfrac{1}{2}\alpha^2\upsilon\rho\mathscr{E}_s \\ &= (4\eta_r(1-\eta_r))\,(\upsilon\,Q_r)\left(\tfrac{1}{8}\omega u\right)\alpha^2 \\ &= (4\eta_r(1-\eta_r))\,\Upsilon\,p\,\alpha^2 \end{aligned} \tag{22}$$

Note that the power delivered to the load may be written as a product of four terms. The first term, $4\eta_r(1-\eta_r)$, is a dimensionless scaling factor which depends only on load efficiency. This scaling factor is plotted in Figure 1.

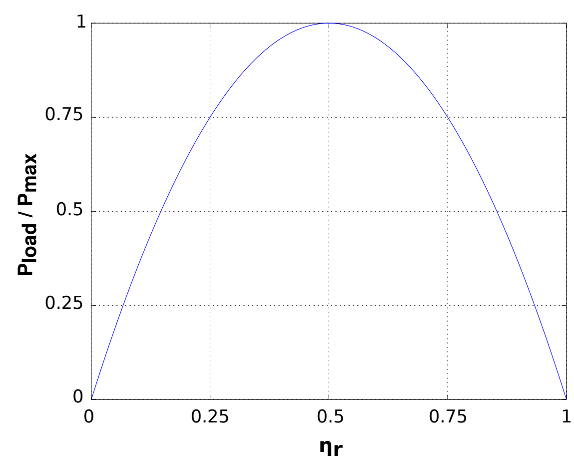

Fig. 1. Ratio between the power transferred to the load and the maximum receivable power as a function of receiver efficiency, η_r, for $\alpha = 1$.

The second term, Υ, is defined to be:

$$\Upsilon \equiv Q_r\upsilon \tag{23}$$

which is the product of the quality factor of the dipole resonator and its dipole volume. Since Υ has units of volume, let it be called the capture volume of the dipole receiver.

The third term, p, may be defined to be:

$$p \equiv \tfrac{1}{8}\omega u \tag{24}$$

Since p has units of power per unit volume, let it be called the power density of the ambient field.

The fourth term, α^2, is simply the square of the dimensionless alignment factor.

The maximum power, P_{max}, which may be delivered to the load occurs at $\eta_r = 1/2$ and $\alpha = 1$, and is given by the product of the capture volume, Υ, and the local power density, p:

$$P_{\text{max}} = \Upsilon\,p \tag{25}$$

V. Optimal Efficiency for Multiple Receivers

Extending the treatment in [1] and [2], the optimal efficiency of power transfer may be calculated as follows. Assume the system contains N resonant dipole receivers which couple to the source resonator, but are sufficiently separated from each other that the coupling between receivers is negligible.[1] Let κ_i denote the coupling coefficient, Q_i denote the unloaded quality factor, and $\mathscr{E}_i$ denote the stored energy of the ith resonant dipole. The total power dissipated in the system through undesired loss mechanisms, $P_{\text{diss-total}}$, is:

$$P_{\text{diss-total}} = P_{\text{diss-s}} + \sum_{i=1}^{N} P_{\text{diss-r}_i} = \frac{\omega \mathscr{E}_s}{Q_s} + \sum_{i=1}^{N} \frac{\omega \mathscr{E}_i}{Q_i} \tag{26}$$

and the total power delivered to all of the loads is:

$$P_{\text{load-total}} = \sum_{i=1}^{N} P_{\text{load}_i} = \sum_{i=1}^{N} \frac{\omega \mathscr{E}_i}{Q_{\text{eff}_i}} \tag{27}$$

Define the incremental strong-coupling parameter, γ_i, to be:

$$\gamma_i \equiv \kappa_i^2 Q_s Q_i \tag{28}$$

In terms of these dimensionless parameters, the efficiency of power transfer, η, may be written as:

$$\eta = \frac{P_{\text{load-total}}}{P_{\text{load-total}} + P_{\text{diss-total}}} = \frac{\sum_i \gamma_i \eta_i (1-\eta_i)}{1 + \sum_i \gamma_i (1-\eta_i)} \tag{29}$$

where η_i is the efficiency of the ith receiver, as defined in Equation 20.

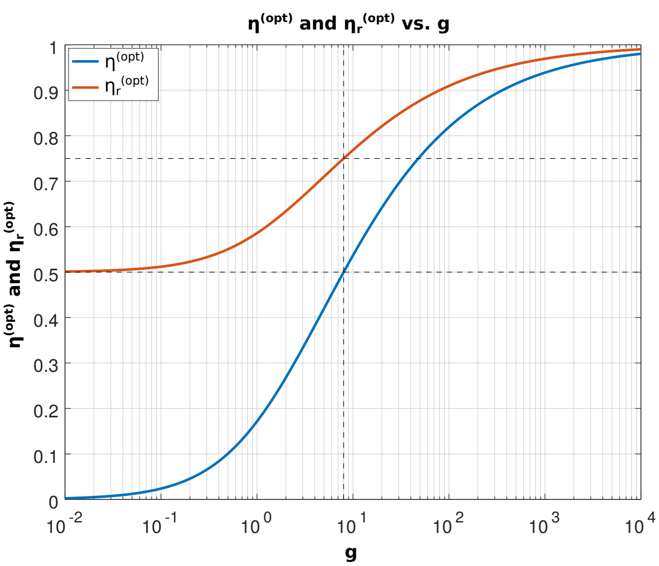

Fig. 2. Optimal system efficiency, $\eta^{(\text{opt})}$, and optimal receiver efficiency, $\eta_r^{(\text{opt})}$, as a function of the strong-coupling parameter, g.

The receiver efficiencies, η_i, may be freely chosen by varying the manner in which the loads couple to the resonant receivers. Therefore, in order to optimize the power transfer, we wish to find the values of η_i which maximize η. If we differentiate $\ln \eta$ with respect to η_j, we get:

$$\begin{aligned}\frac{\partial}{\partial \eta_j} \ln \eta &= \frac{\gamma_j (1-2\eta_j)}{\sum_i \gamma_i \eta_i (1-\eta_i)} - \frac{\gamma_j}{1+\sum_i \gamma_i (1-\eta_i)} \\ &= \gamma_j \frac{(1-2\eta_j) S_2 + S_1}{S_1 S_2}\end{aligned} \tag{30}$$

where the sums, S_1 and S_2, are defined to be:

$$S_1 \equiv \sum_{i=1}^{N} \gamma_i \eta_i (1-\eta_i) \tag{31}$$

$$S_2 \equiv 1 + \sum_{i=1}^{N} \gamma_i (1-\eta_i) \tag{32}$$

To extremize η, we wish to find a set of values for η_j such that the right-hand side of Equation 30 is zero for every value of j:

$$(1-2\eta_j) S_2 + S_1 = 0 \tag{33}$$

Note that because S_1 and S_2 are independent of j, all values of η_j must be equal to a constant that is independent of j. Define η_r to be this constant:

$$\eta_j = \eta_r \tag{34}$$

Equation 34 allows us to simplify the sums in Equations 31 and 32:

$$S_1 = g\, \eta_r (1-\eta_r) \tag{35}$$

$$S_2 = 1 + g\,(1-\eta_r) \tag{36}$$

where the quantity, g, is called the strong-coupling parameter, and is defined to be:

$$g \equiv \sum_i \gamma_i = \sum_i \kappa_i^2 Q_s Q_i \tag{37}$$

Equations 35 and 36 may be substituted into Equation 33 , which yields a quadratic with two solutions for η_r:

$$\eta_r^{\pm} = \frac{(1+g) \pm \sqrt{1+g}}{g} \tag{38}$$

The (+) solution is always greater than one, and the (-) solution is always less than one but greater than zero. Because $0 < \eta_r < 1$, the only valid solution for the optimum receiver efficiency, $\eta_r^{(\text{opt})}$, is:

$$\eta_j = \eta_r^{(\text{opt})} = \eta_r^{-} = \frac{(1+g) - \sqrt{1+g}}{g} = \frac{\sqrt{1+g}}{1+\sqrt{1+g}} \tag{39}$$

We may plug Equations 37 and 39 into Equation 29 to find the optimal system efficiency, $\eta^{(\text{opt})}$:

$$\eta^{(\text{opt})} = \frac{\sqrt{1+g} - 1}{\sqrt{1+g} + 1} \tag{40}$$

Note that there exists a simple relation between the optimal receiver efficiency, $\eta_r^{(\text{opt})}$, and the optimal system efficiency, $\eta^{(\text{opt})}$:

$$\eta_r^{(\text{opt})} = \frac{1+\eta^{(\text{opt})}}{2} \tag{41}$$

The relation between $\eta^{(\text{opt})}$ and g is plotted in Figure 2. Note that $\eta^{(\text{opt})} = 1/2$ and $\eta_r^{(\text{opt})} = 3/4$ when $g = 8$.

[1]The coupling between resonators is negligible as long as each resonator lies far outside the spheres of influence of all the other resonators. See Appendix A for the definition of the size of the sphere of influence of a dipole resonator.

VI. The Dipole Volume of a Circular Loop Receiver

Consider a resonant magnetic dipole receiver consisting of N circular turns of wire. Let D denote the diameter of the loop, and let it be assumed that the wire has a circular cross-section with diameter d. The inductance of the loop is: [3]

$$L = \mu N^2 \frac{D}{2}\left(\ln\left(\frac{8D}{d}\right) - 2\right) \tag{42}$$

which is valid for $D \gg d$. Suppose the loop carries a current, I. The magnetic dipole moment of the loop is:

$$m = \pi\left(\frac{D}{2}\right)^2 NI \tag{43}$$

The dipole volume of the loop is:

$$\upsilon = \frac{\mu m^2}{\mathscr{E}} = \frac{\mu m^2}{\frac{1}{2}LI^2} = \frac{\pi^2 D^3}{4(\ln(8D/d) - 2)} \tag{44}$$

Note that the dipole volume is independent of the number of turns, N.

VII. Effective Source Resonator Volume and Receiver Space-Filling Efficiency

Consider a resonator which fills a certain volume of space, $\mathcal{V}_p$, with a power density, p. Dipole receivers are free to move about this space and receive power. Let ρ_{min} be the minimum of the normalized energy density experienced by any receiver within the pre-defined volume. The source volume, $\mathcal{V}_s$, may be defined to be:

$$\mathcal{V}_s \equiv \frac{1}{\rho_{\text{min}}} = \upsilon_d \mathcal{V}_p \tag{45}$$

where υ_d is a dimensionless factor called the dilution factor. Because $\rho_{\text{min}}\mathcal{V}_p \leq 1$, this implies $\upsilon_d = \mathcal{V}_s/\mathcal{V}_p \geq 1$.

If the field is approximately uniform in the region of interest, ρ may be approximated as a constant:

$$\rho \approx \frac{1}{\mathcal{V}_s} \tag{46}$$

The incremental strong-coupling parameter, γ_i, for the ith receiver, is:

$$\gamma_i = \frac{1}{2}\alpha_i^2 \upsilon_i \rho_i Q_s Q_i \approx \frac{1}{2}\alpha_i^2 \Upsilon_i \frac{Q_s}{\mathcal{V}_s} \tag{47}$$

The strong-coupling parameter, g, is therefore given by:

$$g = \sum_{i=1}^{N} \gamma_i \approx \frac{1}{2} Q_s \frac{\sum_{i=1}^{N} \alpha_i^2 \Upsilon_i}{\mathcal{V}_s} \tag{48}$$

Define the quantity, $\mathcal{V}_c$, to be the total capture volume of all of the receivers, weighted by the square of each receiver's alignment factor:

$$\mathcal{V}_c \equiv \sum_{i=1}^{N} \alpha_i^2 \Upsilon_i \tag{49}$$

The strong-coupling parameter can therefore be written as a function of the space-filling efficiency of the capture volumes of the receivers:

$$g \approx \frac{1}{2} Q_s \frac{\mathcal{V}_c}{\mathcal{V}_s} = \frac{Q_s}{2\upsilon_d}\frac{\mathcal{V}_c}{\mathcal{V}_p} \tag{50}$$

VIII. Idling Power

Due to dissipation in the source resonator, as well as other loss mechanisms, the source resonator will require a certain amount of power, P_{idle}, to maintain its field even in the absence of any receivers. The dissipation in the source resonator provides a lower bound to this idling power:

$$P_{\text{idle}} > P_{\text{diss-s}} = \frac{\omega \mathscr{E}_s}{Q_s} \tag{51}$$

Define $p_{\text{min}} = \omega\, u_{\text{min}}/8$ to be the minimum power density within the pre-defined volume. The lower bound on idling power may be written as:

$$P_{\text{idle}} > \frac{\omega}{Q_s}\frac{u_{\text{min}}}{\rho_{\text{min}}} = \frac{8}{Q_s} p_{\text{min}} \mathcal{V}_s = \frac{8\upsilon_d}{Q_s} p_{\text{min}} \mathcal{V}_p \tag{52}$$

Therefore, the lower bound on the idling power is proportional to the power density multiplied by the volume of the region being powered, and inversely proportional to the quality factor of the source resonator.

IX. Summary of Quantities

The most important quantities derived and defined in the previous sections are summarized in the table below.

Symbol	Name	Definition
α	alignment factor	$\lvert\cos\theta\rvert$
υ	dipole volume	$\frac{\mu\lvert\vec{m}\rvert^2}{\mathscr{E}_r}$ or $\frac{\lvert\vec{p}\rvert^2}{\epsilon\mathscr{E}_r}$
υ_{max}	dipole volume upper bound	$9\mathcal{V}_e$
η_g	geometric efficiency	$\upsilon/\upsilon_{\text{max}}$
Υ	capture volume	$Q_r\,\upsilon$
u	energy density	$\frac{\lvert\vec{B}\rvert^2}{2\mu}$ or $\frac{1}{2}\epsilon\lvert\vec{E}\rvert^2$
p	power density	$\frac{1}{8}\omega u$
ρ	normalized energy density	$\frac{u}{\mathscr{E}_s}$
η_r	receiver efficiency	$P_{\text{load}}/P_{\text{trans}}$
$\mathcal{V}_i$	volume of influence	$\frac{1}{3}(1-\eta_r)\,\alpha\Upsilon$
P_{max}	maximum receivable power	Υp
P_{load}	load power	$4\eta_r(1-\eta_r)\alpha^2 P_{\text{max}}$
γ_i	incremental strong-coupling parameter	$\frac{1}{2}\alpha_i^2\Upsilon_i\rho_i Q_s$
g	strong-coupling parameter	$\sum_{i=1}^{N}\gamma_i$
$\eta^{(\text{opt})}$	optimal system efficiency	$\frac{\sqrt{1+g}-1}{\sqrt{1+g}+1}$
$\eta_r^{(\text{opt})}$	optimal receiver efficiency	$\frac{1+\eta^{(\text{opt})}}{2}$
$\mathcal{V}_c$	total capture volume	$\sum_{i=1}^{N}\alpha_i^2\Upsilon_i$
$\mathcal{V}_s$	source volume	$\frac{1}{\rho_{\text{min}}}$
υ_d	dilution factor	$\mathcal{V}_s/\mathcal{V}_p$
P_{idle}	idling power (lower bound)	$\frac{8\upsilon_d}{Q_s}p_{\text{min}}\mathcal{V}_p$

References

[1] André Kurs, Aristeidis Karalis, Robert Moffatt, J. D. Joannopoulos, Peter Fisher, Marin Soljačić, "Wireless Power Transfer via Strongly Coupled Magnetic Resonances," *Science*, Volume 317, Issue 5834, pp. 83-86, July 2007.

[2] Andre Kurs. (2007) *Power Transfer Through Strongly Coupled Resonances*. (Master's Thesis) http://dspace.mit.edu/handle/1721.1/45429

[3] Frederick W. Grover, *Inductance Calculations*. Mineola, New York: Dover Publications, Inc., 1946.

[4] Mark A. Kemp, Matt Franzi, Andy Haase, Eric Jongewaard, Matthew T. Whittaker, Michael Kirkpatrick, Robert Sparr, "A high Q piezoelectric resonator as a portable VLF transmitter," *Nature Communications*, Volume 10, Article number: 1715, April 2019.

[5] John David Jackson. *Classical Electrodynamics*, 3rd Ed. John Wiley & Sons, Inc. 1999.

Appendix

A. The Sphere of Influence of a Resonant Dipole Receiver

A resonant dipole receiver creates its own electric or magnetic field proportional to its dipole moment. At a given distance, the field is strongest along the axis of the dipole moment vector, and is given by:

$$|\vec{B}_d(r)| = \frac{\mu}{2\pi}\frac{|\vec{m}|}{r^3} \tag{53}$$

for a magnetic dipole and by:

$$|\vec{E}_d(r)| = \frac{1}{2\pi\epsilon}\frac{|\vec{p}|}{r^3} \tag{54}$$

for an electric dipole, where r is the distance from the center of the dipole.

The radius of influence can be defined as the radius, r_i, at which the strength of the self-field of the dipole is equal to the strength of the ambient field:

$$|\vec{B}_d(r_i)| = |\vec{B}| \quad \text{or} \quad |\vec{E}_d(r_i)| = |\vec{E}| \tag{55}$$

From Equations 9, 12, 53, and 55:

$$\frac{|\vec{B}_d(r_i)|^2}{|\vec{B}|^2} = \frac{1}{2(2\pi)^2 r_i^6}\frac{2\mu}{|\vec{B}|^2}\mu|\vec{m}|^2 = \frac{1}{8\pi^2 r_i^6}\frac{\mathscr{E}_r \upsilon}{u} = 1 \tag{56}$$

From Equations 10, 13, 54, and 55:

$$\frac{|\vec{E}_d(r_i)|^2}{|\vec{E}|^2} = \frac{1}{2(2\pi)^2 r_i^6}\frac{2}{\epsilon|\vec{E}|^2}\frac{|\vec{p}|^2}{\epsilon} = \frac{1}{8\pi^2 r_i^6}\frac{\mathscr{E}_r \upsilon}{u} = 1 \tag{57}$$

From Equations 14, 18, 19, 21, and 23:

$$\begin{aligned}\mathscr{E}_r &= \kappa^2 Q_L^2 \mathscr{E}_s = \frac{1}{2}\alpha^2 \upsilon\, Q_r^2 (1-\eta_r)^2 \rho\, \mathscr{E}_s \\ &= \frac{1}{2}\alpha^2 \frac{\Upsilon^2}{\upsilon}(1-\eta_r)^2 u\end{aligned} \tag{58}$$

Which implies:

$$\frac{\mathscr{E}_r \upsilon}{u} = \frac{1}{2}\alpha^2 \Upsilon^2 (1-\eta_r)^2 \tag{59}$$

Therefore, for both electric and magnetic dipoles, the radius of influence is defined by the relation:

$$\frac{\alpha^2\Upsilon^2(1-\eta_r)^2}{16\pi^2 r_i^6} = 1 \quad \Longrightarrow \quad \frac{\alpha\Upsilon(1-\eta_r)}{4\pi r_i^3} = 1 \tag{60}$$

Define the sphere of influence to be the sphere of radius r_i centered on the dipole. The volume of the sphere of influence, $\mathcal{V}_i$, is given by:

$$\mathcal{V}_i = \frac{4\pi}{3} r_i^3 = \frac{1}{3}(1-\eta_r)\,\alpha\Upsilon \tag{61}$$

The receiver power is maximized at $\alpha = 1$, and $\eta_r = 1/2$, in which case $\mathcal{V}_i = {}^1\!/_6\Upsilon$. The maximum volume of influence occurs when the receiver is unloaded and $\alpha = 1$, in which case $\eta_r = 0$, and $\mathcal{V}_i = {}^1\!/_3\Upsilon$.

If a dipole resonator is placed within the sphere of influence of a second dipole resonator, then it is possible for the local field experienced by the first resonator to be dominated by the dipole field of the second resonator rather than the ambient field. Therefore, in order to ensure that interactions between resonators are negligible, each resonator must lie far outside the spheres of influence of all of the other resonators.

B. Upper Bound on Capture Volume

Consider a dipole resonator oscillating at angular frequency, ω. The oscillation causes the dipole resonator to produce far-field dipole radiation. If the dipole is magnetic, the radiated power is:

$$\begin{aligned}P_{\text{rad}} &= \frac{\zeta}{12\pi}\left(\frac{\omega}{c}\right)^4 |\vec{m}|^2 = \frac{\zeta}{12\pi}\left(\frac{\omega}{c}\right)^4 \frac{\upsilon\mathscr{E}_r}{\mu} \\ &= \frac{c}{12\pi}\left(\frac{\omega}{c}\right)^4 \upsilon\mathscr{E}_r\end{aligned} \tag{62}$$

where $\zeta = \sqrt{\mu/\epsilon}$ is the impedance of the surrounding medium, and $c = 1/\sqrt{\epsilon\mu}$ is the speed of light in the surrounding medium. If the dipole is electric, the radiated power is:

$$\begin{aligned}P_{\text{rad}} &= \frac{c^2\zeta}{12\pi}\left(\frac{\omega}{c}\right)^4 |\vec{p}|^2 = \frac{c^2\zeta}{12\pi}\left(\frac{\omega}{c}\right)^4 \upsilon\epsilon\mathscr{E}_r \\ &= \frac{c}{12\pi}\left(\frac{\omega}{c}\right)^4 \upsilon\mathscr{E}_r\end{aligned} \tag{63}$$

The quality factor due to radiation, Q_{rad}, is therefore given by:

$$Q_{\text{rad}} \equiv \frac{\omega\mathscr{E}_r}{P_{\text{rad}}} = \frac{12\pi}{(\omega/c)^3\upsilon} = \frac{3}{2\pi^2}\frac{\lambda^3}{\upsilon} \tag{64}$$

where λ is the wavelength of a plane wave with angular frequency ω.

Let Q_{other} represent the quality factor due to all other loss mechanisms. The overall quality factor, Q_r, of the resonator is therefore given by:

$$\frac{1}{Q_r} = \frac{1}{Q_{\text{rad}}} + \frac{1}{Q_{\text{other}}} > \frac{1}{Q_{\text{rad}}} \tag{65}$$

which implies that:

$$Q_r < Q_{\text{rad}} \tag{66}$$

This gives the following upper bound on the capture volume:

$$\Upsilon = \upsilon Q_r < \upsilon Q_{\text{rad}} = \frac{3}{2\pi}\lambda^3 \tag{67}$$

Therefore, the maximum possible capture volume of any dipole resonator, Υ_{max}, is given by:

$$\Upsilon_{\text{max}} = \frac{3}{2\pi}\lambda^3 \tag{68}$$

C. Antenna Efficiency of an Electrically Small Dipole Antenna

Although it is not directly relevant to near-field wireless power transfer, the quantities derived in the previous section are useful for characterizing the effectiveness of a small dipole resonator as an antenna for either receiving or transmitting radiation, such as the piezoelectric antenna described in [4].

If power is injected into the dipole resonator, some fraction will go to heat, and some fraction will go to far-field radiation:

$$P_{\text{in}} = P_{\text{rad}} + P_{\text{heat}} \tag{69}$$

The antenna efficiency, η_a, of the dipole resonator is defined to be:

$$\eta_a \equiv \frac{P_{\text{rad}}}{P_{\text{in}}} = \frac{P_{\text{rad}}}{P_{\text{rad}} + P_{\text{heat}}} = \frac{\frac{\omega \mathscr{E}_r}{Q_{\text{rad}}}}{\frac{\omega \mathscr{E}_r}{Q_r}} = \frac{Q_r}{Q_{\text{rad}}} = \frac{\upsilon Q_r}{\upsilon Q_{\text{rad}}} \tag{70}$$

Therefore, the antenna efficiency is given by the ratio of the capture volume of the resonator, Υ, to the theoretical maximum capture volume, Υ_{max}:

$$\eta_a = \frac{\Upsilon}{\Upsilon_{\text{max}}} \tag{71}$$

D. Absolute Upper Bound on Dipole Volume

Consider a resonator containing a compact charge and current distribution which is fully enclosed within a sphere of radius, R, and volume, $\mathcal{V}_e = 4\pi R^3/3$. Let it be assumed that the medium outside the sphere has uniform permeability, μ, and permittivity, ϵ.

The energy of the resonator, $\mathscr{E}$, will have contributions from both the interior and exterior of the sphere:

$$\mathscr{E} = \mathscr{E}_{\text{int}} + \mathscr{E}_{\text{ext}} \tag{72}$$

The exterior energy will be given by the integral of the energy density, u, over the space outside the sphere:

$$\mathscr{E}_{\text{ext}} = \int_{r>R} u \, d\mathcal{V} \tag{73}$$

where r is the distance from the origin. The electric and magnetic fields outside of the sphere may be written as a sum of multipole fields. Because the multipole fields are orthogonal functions, their contributions to the integral are all independent and non-negative. This implies that the energy contribution from the dipole field by itself must give a lower bound on the energy of the exterior, which gives a lower bound on the energy of the resonator. The integrated energy, $\mathscr{E}_m$, of a magnetic dipole field over the region $r > R$ is:

$$\mathscr{E}_m = \frac{\mu |\vec{m}|^2}{9\mathcal{V}_e} \tag{74}$$

and the integrated energy, $\mathscr{E}_p$, of an electric dipole field over the region $r > R$ is:

$$\mathscr{E}_p = \frac{|\vec{p}|^2}{9\epsilon \mathcal{V}_e} \tag{75}$$

This gives the following upper bound for the dipole volumes of magnetic and electric dipole resonators:

$$\upsilon_m = \frac{\mu|\vec{m}|^2}{\mathscr{E}} \leq \frac{\mu|\vec{m}|^2}{\mathscr{E}_{\text{ext}}} \leq \frac{\mu|\vec{m}|^2}{\mathscr{E}_m} = 9\mathcal{V}_e \tag{76}$$

$$\upsilon_p = \frac{|\vec{p}|^2}{\epsilon\mathscr{E}} \leq \frac{|\vec{p}|^2}{\epsilon\mathscr{E}_{\text{ext}}} \leq \frac{|\vec{p}|^2}{\epsilon\mathscr{E}_p} = 9\mathcal{V}_e \tag{77}$$

Therefore, in general, the following upper bound must hold for the dipole volume of any type of resonator:

$$\upsilon \leq 9\mathcal{V}_e \tag{78}$$

where $\mathcal{V}_e$ is the volume of the smallest sphere which completely encloses the resonator.

E. Dipole Volume of a Uniformly Magnetized Sphere

Suppose that a sphere of volume, $\mathcal{V}_e$, is filled with a permeable material, with relative permeability, μ_r, which is uniformly magnetized by means of a surface current. According to Jackson [5], the magnetic field is uniform inside the sphere, and takes the form of a pure dipole field outside the sphere. Adding the energy contributions from the interior and exterior gives the following dipole volume:

$$\upsilon_m = \frac{\mu|\vec{m}|^2}{\mathscr{E}_{\text{int}} + \mathscr{E}_{\text{ext}}} = \frac{9}{1 + 2/\mu_r}\mathcal{V}_e \tag{79}$$

Note that as $\mu_r \to \infty$, the dipole volume of the magnetic resonator approaches the upper bound given by Condition 78.

F. Stricter Upper Bounds on Dipole Volume Assuming a Homogeneous Medium

Consider the dipole resonator consisting of a sphere with a uniform magnetic field on its interior, as discussed in the previous section. If the relative permeability is set to 1, the medium is homogeneous everywhere. In that case, according to Equation 79, the dipole volume of the magnetic resonator becomes $3\mathcal{V}_e$.

In general, it can be shown that the dipole volume of any current distribution must satisfy the following constraint when embedded in a medium which is homogeneous everywhere:

$$\upsilon_m \leq 3\mathcal{V}_e \tag{80}$$

Constraint 80 may be proven by calculus of variations. The energy may be minimized over all possible current distributions completely enclosed by a sphere of volume, $\mathcal{V}_e$, and constrained to have a fixed dipole moment, $\vec{m}$. Out of all such current distributions, the lowest energy is achieved by the current distribution which creates a uniform magnetic field on the interior of the sphere.

The same calculus of variations may be applied to the case of an electric dipole resonator in a homogeneous medium. In that case, the lowest energy is achieved by the charge distribution which creates a uniform electric field on the interior of the sphere. However, different boundary conditions cause the interior field to contain less energy, giving the following upper bound on dipole volume for an electric dipole resonator in a medium which is homogeneous everywhere:

$$\upsilon_p \leq 6\mathcal{V}_e \tag{81}$$

Note that these upper limits on dipole volume provide a lower bound on the quality factor due to radiation described by Equation 64.